Nonlinear Dynamic Buckling of Discrete Dissipative or Nondissipative Systems under Step Loading

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Nonlinear dynamic buckling of nonlinearly elastic, nondissipative or dissipative, multimass systems under step loading of infinite or finite duration is thoroughly discussed. Attention is restricted to discrete structural systems that under the same loading applied statically experience snapping or bifurcational instability associated with an unstable branching point lying either on a trivial or a nontrivial primary equilibrium path. Considering the stability of motion in the large the mechanism of dynamic instability and the nature of the dynamic critical point are properly investigated. Dynamic buckling criteria leading to exact dynamic buckling loads, as well as a simple static buckling criterion yielding lower bound buckling estimates of practical importance, are established. Finally, error estimates of the lower bound buckling loads obtained by the static buckling criterion are also assessed.

Introduction

THE conditions under which imperfection-sensitive systems under dynamic loading experience an unbounded motion, becoming dynamically unstable, have been discussed in some early studies referring basically to structures that can be treated as one-mass systems.[1-4] Budiansky and Hutchinson,[5] Hutchinson and Budiansky,[6] as well as Budiansky,[7] presented a thorough extension of Koiter's postbuckling analysis of the dynamic buckling of imperfection-sensitive single-degree-of-freedom undamped systems under step loading of infinite or finite duration (of rectangular or triangular shape); namely, systems that under the same loading applied statically exhibit snapping. Other interesting studies worth mentioning have been presented by Humphreys[8-10] and Thompson.[11] Further contributions in this area of research, referring mainly to one-mass systems, have been presented by Handelman,[12] Kounadis,[13] Simitses, Kounadis and Giri,[14] Elishakoff,[15] Holasut and Ruiz,[16] Kounadis and Mallis,[17] and an excellent review of the subject was given by Simitses.[18] Recently, this author—in conjunction with Raftoyiannis,[19,20] Mahrenholtz and Bogacz,[21] and Raftoyiannis and Mallis[22]—presented a comprehensive dynamic buckling analysis according to which the static stability criterion associated with zero total potential energy yields 1) exact dynamic buckling loads only in the case of single-degree-of-freedom systems, if damping is ignored and 2) lower bound dynamic buckling estimates, when viscous damping is included. Moreover, it has been shown[20] that the dynamic instability criterion associated with zero phase point velocity is of general validity for one-degree-of-freedom nondissipative or viscous dissipative systems under step loading of infinite duration. A few numerical results, based on a particular two-degree-of-freedom (damped or undamped) system under step loading of infinite duration, indicate that the aforementioned dynamic and static instability criteria could be extended after suitable adjustment to multimass systems.[23,24]

In this investigation, two nonlinear two-degree-of-freedom systems with or without damping under step loading of infinite or finite duration are used as models. A theoretical analysis of these models combined with a variety of numerical results is aimed at the following objectives:

1) To clarify the mechanism of dynamic instability and the nature of dynamic critical points considering the stability of motion in the large in the sense of Lagrange.

2) To establish criteria leading to exact dynamic buckling loads.

3) To determine simple static criteria yielding lower bound dynamic buckling loads of practical importance without solving the highly nonlinear differential equations of motion.

4) To estimate the maximum possible error of the foregoing lower bound dynamic buckling loads.

In this study, in addition to limit point systems, we consider bifurcational systems with an unstable distinct branching point lying either on trivial or a nontrivial fundamental path.

The integration is achieved numerically using Runge Kutta's fourth-order scheme, as well as in some cases an approximate but efficient analytic technique.[17,25] The significance of approximate techniques and particularly of lower bound dynamic buckling estimates lies in the fact that nonlinear dynamic instability quite often occurs at large time, after a long period of apparent tranquility. This implies considerable computational difficulties.

General Considerations

The dynamic analysis that follows refers to structural systems that are discrete or have been discretized by some approximation technique. Thus, one can consider a general n-degree-of-freedom holonomic nonlinear system[26] under time-dependent loading $\Lambda(t)$ described by a set of generalized displacements q_i and velocities $\dot{q}_i (i = 1, ..., n)$, respectively. Lagrange's equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial U_T}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0, \qquad (i = 1, ..., n) \quad (1)$$

where dots denote differentiation with respect to time t; $K = (1/2)\Sigma\alpha_{ij}\dot{q}_i\dot{q}_j$ is the total kinetic energy with coefficients $\alpha_{ij} = \alpha_{ij}\ (q_1, ..., q_n)$; $U_T = (q_1, ..., q_n;\Lambda)$ is the total potential energy, being a linear function of Λ; $F = (1/2)\Sigma c_{ij}\dot{q}_i\dot{q}_j$ is the Rayleigh dissipation function, with $[c_{ij}]$ being a symmetric nonnegative matrix. The dynamic loading is of the form $\Lambda = \lambda f(t)$, where λ is a constant loading parameter and $f(t)$ the forcing function. Although the latter function may be of any type, particular emphasis is given to the step loading of infinite duration, i.e.,

$$f(t) = 0 \qquad \text{for } t < 0 \quad (2a)$$

$$f(t) = 1 \qquad \text{for } t \geq 0 \quad (2b)$$

In this case $\Lambda(t) = \lambda$, and the system becomes autonomous.[26]

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Considering the system initially, ($t_o = 0$) at rest, we may assume the following initial conditions:

$$q_i(0) = q_i^o, \qquad \dot{q}_i(0) = 0 \tag{3a}$$

which yield

$$K \mid_{t_o=0} = U_T \mid_{t_0=0} = 0 \tag{3b}$$

Then the total energy E, based on Eqs. (1), at any time $t > 0$ is equal to

$$E = K + U_T + 2\int_0^t F \, dt = 0 \tag{4}$$

This equation can be used as a measure for checking the accuracy of numerical results, particularly when large time solutions are required.

The set of n second-order Lagrangian [Eqs. (1)] can be replaced by a system of $2n$ first-order Hamiltonian equations.[26] This system, subject to the initial conditions (3), can be written in the form

$$\left.\begin{array}{l} \dot{y}_i = Y_i(y_1, ..., y_{2n}; \lambda) \\ y_i(t_o) = C_i(y_1^o, ..., y_n^o; \underbrace{0, ..., 0}_{n}) \end{array}\right\} i = 1,2, ..., 2n \tag{5}$$

Equations (5) can be written in matrix-vector form, as follows:

$$\dot{y} = Y(y;\lambda), \qquad y(t_o) = C \tag{5$'$}$$

where y is the state vector being continuously dependent on t and λ, and the nonlinear vector function Y is assumed to satisfy the Lipschitz conditions in a given domain D. This implies also that $q_i(t)$ belongs to the class of functions C_2 (i.e., $q_i(t) \epsilon C_2$). According to the Cauchy-Lipschitz theorem, the initial-value problem of Eqs. (5), or Eq. (5′), is uniquely determined by the initial conditions[27] and is equivalent to the solution of the integral equation

$$y(t) = C + \int_{t_o}^t Y[y(s),s] \, ds \tag{6}$$

On the basis of Eq. (6), the following approximations can be established:

$$y_{(r+1)}(t) = C + \int_{t_o}^t Y[y_r(s),s] \, ds \qquad (r = 1,2,...) \tag{7}$$

The convergence of this iteration scheme can be substantially improved by a modified procedure proposed by the author.[17,25] This procedure is used as an auxiliary technique for checking the accuracy of some of the numerical results based on the Runge-Kutta scheme, which is mainly employed herein.

The equilibrium (singular) states y^E are obtained by the equation

$$Y(y^E;\lambda) = 0 \tag{8}$$

whereas for ordinary or regular (nonequilibrium) points $Y^T Y > 0$, where Y^T is the transpose of Y.

For the stability of an equilibrium state y^E, we examine the motion in its neighborhood by superimposing the disturbance ξ to y^E. Inserting $y = y^E + \xi$ into the first of Eq. (5′) and expanding $Y(y^E + \xi;\lambda)$ in a Taylor series around y^E, we obtain the following linear variational equation:

$$\dot{\xi} = Y(y^E;\lambda) + Y_y(y^E;\lambda)\xi \tag{9}$$

which, due to Eq. (8), becomes

$$\dot{\xi} = Y_y(y^E;\lambda)\xi \tag{9$'$}$$

where $Y_y(y^E;\lambda) = \partial Y(y^E;\lambda)/\partial y$ is the Jacobian matrix evaluated at the equilibrium state. When all its eigenvalues have negative real parts, y^E is stable or asymptotically stable (when damping is included). On the other hand, if at least one eigenvalue has a positive real part, y^E is unstable. The number of eigenvalues with a positive real part defines the instability index of an equilibrium point. Increasing values of the index correspond to increasing degrees of instability near the equilibrium point. From Eq. (9′), it is clear that the nature of the equilibrium point depends on the loading λ.

In a similar way, we can establish the stability of a nonsingular (nonequilibrium) point. If $y_o = y_o(t)$ is a known nonsingular solution of Eqs. (5′) and $y = y(t)$ is a neighboring solution (being the perturbed motion), then introducing $y = y_o + \xi$ into the first of Eqs. (5′) and taking into account that $\dot{y}_o = Y(y_o;\lambda)$ we obtain the following linearized equation:

$$\dot{\xi} = Y(y_o + \xi;\lambda) - Y(y_o;\lambda) = Y_y(y_o;\lambda)\xi \tag{10}$$

Apparently, the stability or instability of the nonsingular (regular) point depends on the nature of the eigenvalues of the Jacobian matrix $Y_y(y_o;\lambda)$.

Considering the stability of motion in the large, in the sense of Lagrange (boundedness of solution), dynamic buckling is defined as that state for which an unbounded (escaped) motion corresponding to the minimum load λ is possible. Such a state cannot occur unless at least one of the Jacobian matrix eigenvalues has a positive real part. Note that the existence of at least one eigenvalue with positive real part is guaranteed a priori for the structural systems under discussion that are associated with unstable postbuckling paths.

The next section deals with one-degree-of-freedom systems. Although it basically constitutes a brief recapitulation of previous studies, some additional results useful for the purposes of the subsequent analysis are also presented.

One-Degree-of-Freedom Systems

For a single-degree-of-freedom system with mass m and viscous damping coefficient c, Eqs. (1) and (4) are written as follows:

$$m\ddot{q} + c\dot{q} + \frac{dU_T}{dq}(q;\lambda) = 0 \tag{11a}$$

$$\frac{1}{2}m\dot{q}^2 + c\int_{t_o}^t \dot{q}^2 \, dt + U_T(q;\lambda) = 0 \tag{11b}$$

For an imperfection sensitive (or limit point) system, $U_T(q;\lambda)$ is a nonlinear function of q, at least of third-order. The sufficient condition for an unbounded (escaped) motion is satisfied when the system, under the action of the minimum possible load, is forced to pass through an unstable equilibrium configuration.[19,20,23] Such a load is defined as the dynamic buckling load and is equal to λ_D for $c = 0$ and λ_{DD} for $c \neq 0$. Namely, regardless of whether or not damping is included, the dynamic critical point is a singular (equilibrium) point on the unstable post-buckling equilibrium path. It was also shown by Raftoyiannis and Kounadis[19] that the dynamic buckling load (or λ_{DD}) is the smallest load for which

$$\dot{q} = \ddot{q} = 0 \tag{12}$$

that is, the dynamic curve q vs t exhibits an inflection point. The satisfaction of the Lipschitz conditions implies that $q(t) \epsilon C_2$.

When damping is ignored, ($c = 0$) [Eqs. (11)] due to the dynamic instability criterion of Eq. (12) implying

$$\frac{dU_T}{dq} = U_T = 0 \tag{13}$$

Equations (13) constitute the static instability criterion applicable to limit point systems. This criterion yields dynamic buckling loads $\tilde{\lambda}_D$, which coincide with the exact dynamic buckling loads $\lambda_D(\equiv \tilde{\lambda}_D)$ if there is no damping. When damping is included, such a criterion yields lower bound dynamic buckling estimates, i.e., $\tilde{\lambda}_D < \lambda_{DD}$.[19,21] For zero imperfection, the limit point system becomes a perfect unstable bifurcational system with a critical load λ_c. Since at the branching equilibrium point $U_T|^c = 0$, Eqs. (13) yield $\tilde{\lambda}_D \equiv \lambda_D$. In view of the above development, it follows that $\tilde{\lambda}_D \equiv \lambda_D \equiv \lambda_{DD} \equiv \lambda_c$.

Note that the inflection point dynamic instability criterion (12) can also be derived from physical considerations. In a bounded motion, the trajectories are closed; the maximum q occurs for some $t = t^*$ for which $\dot{q}(t^*) = 0$ with $\ddot{q}(t) < 0$ (implying that the motion changes sense). The motion does not change sense (escaped motion becoming unbounded for $t \to \infty$) when for some $t = t_{cr}$ we have

$$\dot{q}(t_{cr}) = \ddot{q}(t_{cr}) = 0 \tag{14}$$

which hold regardless of whether or not damping is accounted for. This criterion is also valid for other shapes of forcing functions. Note that for rectangular or triangular shape (finite duration) loads there appear discontinuities in $\ddot{q}$ (acceleration) at the instant the load is removed or vanishes. Since in both cases dynamic buckling occurs after the discontinuity, where $q(t)\epsilon C_2$, conditions (14) are guaranteed and the inflection point dynamic instability criterion is also valid. The Jacobian matrix evaluated at an unstable equilibrium point y^E [for which $d^2U_T(q^E)/dq^2 < 0$]

$$\begin{bmatrix} 0 & m \\ -\dfrac{d^2U_T(q^E)}{dq^2} & -c \end{bmatrix} \tag{15}$$

has one negative and one positive eigenvalue. Such a point coinciding with the dynamic critical displacement $q_D(\equiv q^E)$ is a saddle.

As mentioned previously, when the system becomes imperfect the dynamic critical states (q_D, λ_D) or (q_{DD}, λ_{DD}) correspond to unstable equilibrium points different from the (static) limit point. In view of the preceding development[19,20]

$$\tilde{\lambda}_D \equiv \lambda_D < \lambda_{DD} < \lambda_s \tag{16a}$$

$$|\tilde{q}_D| \equiv |q_D| > |q_{DD}| > |q_s| \tag{16b}$$

where $\lambda_s(<\lambda_c)$ is the limit point load being, therefore, an upper bound dynamic buckling estimate. Inequality [Eq. (16)] allows us to have a measure of the maximum possible error of any lower bound buckling estimate $\tilde{\lambda}_D$ when damping is included. This error is equal to $\lambda_s - \tilde{\lambda}_D$.

Since $\lambda_c \equiv \lambda_D(\equiv \tilde{\lambda}_D \equiv \lambda_{DD})$ from inequality (16), it follows that

$$\lambda_c - \lambda_D > \lambda_c - \lambda_s \tag{17}$$

that is, the imperfection sensitivity is more severe under dynamic (step) loading than under a static one (Fig. 1).

The dynamic buckling load of (unstable) bifurcational systems under step load of finite duration of rectangular or triangular shape increases considerably with the decrease of time of application of loading.[6,20] This effect is more pronounced in the critical displacements. A typical plot for the case of a rectangular loading is shown in Fig. 2. For $t_o \to 0$ it follows $\lambda_D \to \infty$, whereas $t_o \to \infty$ implies λ_D^∞ (step loading of infinite duration).

Subsequently, an attempt is made to extend the foregoing instability criteria to multimass systems.

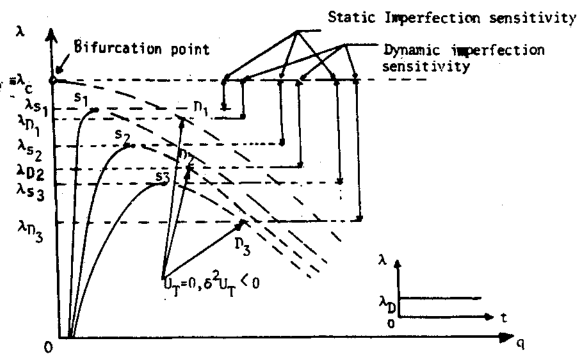

Fig. 1 Static and dynamic imperfection sensitivity for a single-degree-of-freedom undamped system.

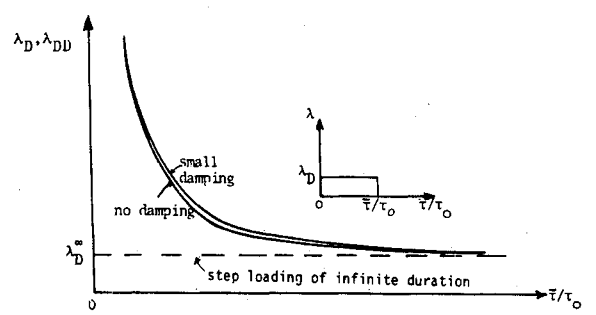

Fig. 2 Dynamic buckling load λ_D (or λ_{DD}) vs time of application of the loading of a single-degree-of-freedom system with (or without) damping.

Multimass Systems

The preceding physical considerations regarding the inflection point instability criterion (which were based on theoretical findings) can be extended to systems with two or more degrees of freedom, as follows: If for sufficiently small values of the step loading $\lambda(<\lambda_D)$, a multimass system undergoes nonlinear vibrations of bounded amplitude, then the maximum value of every generalized displacement $q_i(i = 1, ..., n)$ occurs at some time t^*, for which

$$\dot{q}_i(t^*) = 0, \qquad \ddot{q}_i(t^*) < 0 \tag{18}$$

The last inequality shows that the motion of the mass m_i at the time of maximum displacement changes sense or sign (motion of bounded amplitude).

One can also observe that the maxima of all generalized displacements $q_i(i = 1, ..., n)$ do not take place at the same time (as this occurs in normal modes of linear systems) but, in general, at different times. Moreover, we observe that an unbounded motion could initiate from one generalized displacement, say q_k, for which at some time $t = t_{cr}$ we have

$$\dot{q}_k(t_{cr}) = 0 \qquad \text{with} \qquad \ddot{q}_k(t_{cr}) = 0 \tag{19}$$

that is the inflection point criterion is satisfied.

The last condition assures that the motion of the mass q_k does not change sense. Then an escaped motion (becoming unbounded for $t \to \infty$), which is initiated from at least one generalized coordinate, leads shortly after to a divergent motion of the entire system. The case is somewhat similar with the convergence of an iteration scheme. A vector solution y diverges if one of its components $y_i(i = 1,2, ...)$ diverges.

Application of the aforementioned sufficient dynamic instability criterion presupposes that every $q_i(t)\epsilon C_2$, which is guaranteed if the Lipschitz conditions are satisfied at least near the time when dynamic buckling occurs; this includes the case of step loading of finite duration with various shapes. Certainly, since the criterion of Eqs. (19) is sufficient, an unbounded motion could also occur in a situation when this criterion is not satisfied.

Loading of Infinite Duration

Since condition (19) must be satisfied for at least one k, then in general there will be an i such that $\dot{q}_i \neq 0$ and $\ddot{q}_i \neq 0$ for

$i \neq k$ and, hence, the kinetic energy K does not vanish. Inasmuch as K is a positive definite function with respect to $\dot{q}_i$, Eq. (4) implies that throughout the motion, until the instant of dynamic buckling, U_T is negative definite. Moreover, since $\ddot{q}_i \neq 0$ (for $i \neq k$), the (static) equilibrium equations associated with Eqs. (1) are not satisfied at the instant of dynamic buckling, i.e., $\partial U_T/\partial q_i \neq 0$ unless $i = k$. Therefore, for systems with more than one degree of freedom, the dynamic critical point via which a divergent motion takes place is, in general, a regular (nonsingular) point.

The static instability criterion

$$U_T = \frac{\partial U_T}{\partial q_i} = 0 \qquad (i = 1, \ldots, n) \tag{20}$$

although it is not valid, can be used for obtaining lower bound dynamic buckling estimates $\tilde{\lambda}_D$. As shown above, from the onset of motion until dynamic buckling U_T is negative definite; hence, at the dynamic critical state $(q_1, \ldots, q_n)$ one must have

$$U_T(q_1, \ldots, q_n, \lambda) < 0 \tag{21}$$

Expanding U_T about the (static) equilibrium state $(\tilde{q}_1, \ldots, \tilde{q}_n, \tilde{\lambda}_D)$ obtained through the solution of Eqs. (20) and taking into account that U_T is a linear function of λ, one can write

$$U_T(q_1,\ldots,q_n,\lambda) = U_T(\tilde{q}_1,\ldots,\tilde{q}_n,\tilde{\lambda}_D) + \sum_{i=1}^{n}(q_i - \tilde{q}_i) \left.\frac{\partial U_T}{\partial q_i}\right|_{\substack{q_1 = \tilde{q}_1 \\ \vdots \\ q_n = \tilde{q}_n \\ \lambda = \tilde{\lambda}_D}} + (\lambda - \tilde{\lambda}_D)\left.\frac{\partial U_T}{\partial \lambda}\right|_{\substack{q_1 = \tilde{q}_1 \\ \vdots \\ q_n = \tilde{q}_n}} + \ldots \tag{22}$$

By virtue of relations (20) and (21) and given that $\partial U_T/\partial\lambda < 0$ (being equal to the minus of displacement in the direction of λ) Eq. (22) yields $\lambda > \tilde{\lambda}_D$; that is, the exact dynamic buckling load $\lambda = \lambda_D$ is always higher than the approximate buckling load $\tilde{\lambda}_D$ obtained by solving the system of Eqs. (20), i.e.,

$$\tilde{\lambda}_D < \lambda_D \tag{23}$$

When damping is included, one can show that inequality (16) is still valid, i.e.,

$$\tilde{\lambda}_D < \lambda_D < \lambda_{DD} < \lambda_s \tag{24}$$

That is, for limit point systems with or without damping the maximum possible error in $\tilde{\lambda}_D$ is less than $\lambda_s - \tilde{\lambda}_D$($\lambda_s$ being an upper bound of $\tilde{\lambda}_D$).

For unstable bifurcational systems, two cases will be considered, depending on whether or not the branching point lies on a trivial or a nontrivial fundamental path.

Bifurcation on a Trivial Fundamental Path

In this case Eqs. (20) are identically satisfied and, therefore, $\tilde{\lambda}_D$ coincides with the static bifurcational load λ_c, i.e., $\tilde{\lambda}_D \equiv \lambda_c$. Since, for damped or undamped systems, $\tilde{\lambda}_D$ (based on $U_T = 0$) is a lower bound dynamic buckling load, which for such bifurcational systems coincides with the upper bound dynamic buckling load λ_c, it follows that

$$\tilde{\lambda}_D \equiv \lambda_D \equiv \lambda_{DD} \equiv \lambda_c \tag{25}$$

The effect of imperfection in a nonsymmetric system leads to inequality (24) from which it is evident that $\lambda_c - \lambda_D > \lambda_c - \lambda_s$; namely, the imperfection sensitivity under dynamic loading is more severe than under static loading. These findings extend the relevant conclusions of one-mass systems [e.g., Eq. (17)] to multimass systems.

Bifurcation on a Nontrivial Fundamental Path

This case appears in symmetric systems that are imperfect with axisymmetric imperfections. For such systems with nonlinear fundamental paths, U_T is different from zero at the branching point. However, $U_T = 0$ (being zero at an equilibrium state of the unstable postbuckling path) yields $\tilde{\lambda}_D$, which, in view of the preceding development, satisfies the inequality

$$\tilde{\lambda}_D < \lambda_D < \lambda_{DD} < \lambda_{cc} \tag{26}$$

where λ_{cc} is the bifurcational buckling load of the symmetric (with axisymmetric imperfections) system. If this system becomes slightly nonsymmetric, the branching point degenerates into a limit-point with corresponding static load λ_s satisfying inequality (24). A few numerical results[23,29] indicate that the difference $\lambda_D - \tilde{\lambda}_D$ (or $\lambda_{DD} - \tilde{\lambda}_D$) decreases considerably as the system deviates from its symmetric configuration.

The foregoing findings will be checked below by studying the dynamic stability of two models with two degrees of freedom. With the aid of these models, some new results will be also presented.

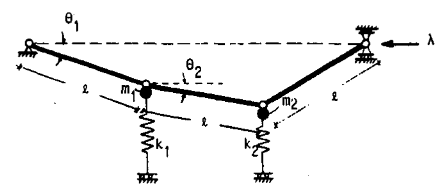

Fig. 3 Geometry of an imperfect two-degree-of-freedom damped system composed of three rigid links interconnected with each other by hinges.

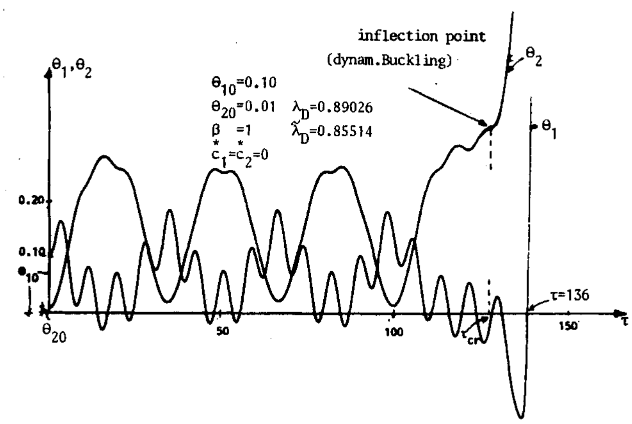

Fig. 4 Unbounded motions Θ_1 and Θ_2 vs τ of an undamped nonsymmetric system.

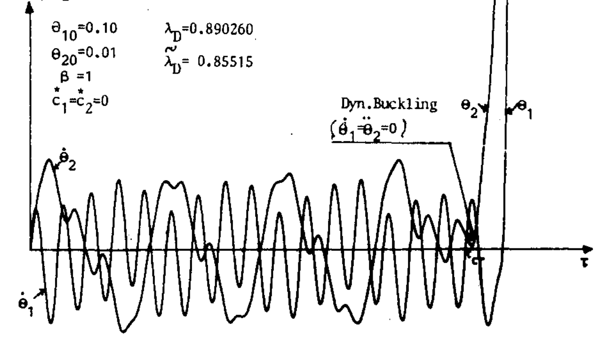

Fig. 5 Angular velocities $\dot{\Theta}_1$ and $\dot{\Theta}_2$ vs τ (near dynamic buckling) of an undamped nonsymmetric system.

Model 1

This is a dissipative nonlinear elastic cubic type structural model,[7] composed of two pin-jointed rigid links of equal length ℓ, which carry two concentrated masses m_1 and m_2 (Fig. 3). The model is subjected to a step load λ of infinite duration applied at its movable end. Denoting by ϑ_{10} and ϑ_{20} the initial angle deviations and by ϑ_1 and ϑ_2 the total angles after deformation (measured from the horizontal), the kinetic energy K, the strain energy U, the potential energy of the external loading Ω, and the Rayleigh dissipation function F, are given by

$$K = \frac{1}{2} m_1 \ell^2 \dot{\vartheta}_1^2 + \frac{1}{2} m_2 \ell^2 [\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1 \dot{\vartheta}_2 \cos(\vartheta_1 - \vartheta_2)] \quad (27a)$$

$$\frac{U}{\ell^2} = \frac{k_1}{2} (\sin\vartheta_1 - \sin\vartheta_{10})^2 - \frac{k_1 \beta_1}{4} (\sin\vartheta_1 - \sin\vartheta_{10})^4 + \frac{k_2}{2} (\sin\vartheta_1 - \sin\vartheta_{10} + \sin\vartheta_2 - \sin\vartheta_{20})^2 - \frac{k_2 \beta_2}{4} (\sin\vartheta_1 - \sin\vartheta_{10} + \sin\vartheta_2 - \sin\vartheta_{20})^4 \quad (27b)$$

$$\frac{\Omega}{\ell} = -\lambda[\cos\vartheta_{10} - \cos\vartheta_1 + \cos\vartheta_{20} - \cos\vartheta_2 + \sqrt{1 - (\sin\vartheta_{10} + \sin\vartheta_{20})^2} - \sqrt{1 - (\sin\vartheta_1 + \sin\vartheta_2)^2}] \quad (27c)$$

$$F = \frac{1}{2} c_1 \ell^2 \dot{\vartheta}_1^2 \cos^2\vartheta_1 + \frac{1}{2} c_2 \ell^2 (\dot{\vartheta}_1 \cos\vartheta_1 + \dot{\vartheta}_2 \cos\vartheta_2)^2 \quad (27d)$$

For the sake of simplicity we may consider the case of symmetric springs by setting $k_1 = k_2 = k$ and $\beta_1 = \beta_2 = \beta$.

Introducing the nondimensionalized quantities

$$\theta(\tau) = \vartheta(t), \qquad \tau = t\sqrt{k/m_2}, \qquad m = m_1/m_2$$

$$\overset{*}{c}_1 = \frac{c_i}{\sqrt{k m_2}} \ (i = 1,2), \qquad \lambda_c = \frac{k\ell}{3}$$

Equations (1) by virtue of relations (27) give

$$(1 + m)\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (\overset{*}{c}_1 + \overset{*}{c}_2)\dot{\theta}_1 \cos^2\theta_1 + \overset{*}{c}_2 \dot{\theta}_2 \cos\theta_1 \cos\theta_2 + \frac{\partial V}{\partial \theta_1} = 0 \quad (28a)$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \overset{*}{c}_2 \dot{\theta}_2 \cos^2\theta_2 + \overset{*}{c}_2 \dot{\theta}_1 \cos\theta_1 \cos\theta_2 + \frac{\partial V}{\partial \theta_2} = 0 \quad (28b)$$

where

$$\frac{\partial V}{\partial \theta_1} = [\sin\theta_1 - \sin\theta_{10} - \beta(\sin\theta_1 - \sin\theta_{10})^3 + (\sin\theta_1 - \sin\theta_{10} + \sin\theta_2 - \sin\vartheta_{20}) - \beta(\sin\theta_1 - \sin\theta_{10} + \sin\theta_2 - \sin\theta_{20})^3] \cos\theta_1 - \frac{\lambda}{3\lambda_c}\left[\sin\theta_1 + \frac{(\sin\theta_1 + \sin\theta_2)\cos\theta_1}{\sqrt{1 - (\sin\theta_1 - \sin\theta_2)^2}}\right] \quad (29a)$$

$$\frac{\partial V_T}{\partial \theta_2} = [\sin\theta_1 - \sin\theta_{10} + \sin\theta_2 - \sin\theta_{20} - \beta(\sin\theta_1 - \sin\theta_{10} + \sin\theta_2 - \sin\theta_{20})^3] \cos\theta_2 - \frac{\lambda}{3\lambda_c}\left[\sin\theta_2 + \frac{(\sin\theta_1 + \sin\theta_2)\cos\theta_2}{\sqrt{1 - (\sin\theta_1 + \sin\theta_2)^2}}\right] \quad (29b)$$

Table 1 Exact and approximate dynamic buckling loads λ_D (or λ_{DD}) and $\tilde{\lambda}_D$, as well as static buckling loads λ_s or λ_{cc}, for various values of Θ_{10}, Θ_{20} and $\overset{*}{c}$

Θ_{10}	Θ_{20}	$\overset{*}{c}$	λ_D or λ_{DD}	$\tilde{\lambda}_D$	λ_{cc} or λ_s	$(\lambda_D - \tilde{\lambda}_D)/\lambda_D \times 100$
0.05	0.00	0.00	0.99214	0.95255	0.99532	3.99
		0.10	0.99532			
	0.01	0.00	0.90413	0.89160	0.92212	1.38
		0.10	0.91516			
	0.02	0.00	0.85423	0.85569	0.88069	0.99
		0.10	0.86758			
	0.05	0.00	0.74891	0.74369	0.79063	0.70
		0.10	0.76493			
	0.10	0.00	0.62965	0.62636	0.68284	0.52
		0.10	0.64548			
	0.20	0.00	0.46904	0.46658	0.52816	0.51
		0.10	0.48229			
	0.40	0.00	0.26928	0.26851	0.32014	0.28
		0.10	0.27611			
0.10	0.00	0.00	0.97102	0.90191	0.98137	7.11
		0.10	0.98118			
	0.01	0.00	0.89026	0.85514	0.90895	3.94
		0.10	0.90194			
	0.02	0.00	0.84061	0.81584	0.86768	2.94
		0.10	0.85452			
	0.05	0.00	0.73474	0.72234	0.77775	1.69
		0.10	0.75155			
	0.10	0.00	0.61464	0.60943	0.67007	0.85
		0.10	0.63226			
	0.20	0.00	0.45623	0.45362	0.51586	0.57
		0.10	0.47045			
	0.40	0.00	0.25953	0.25904	0.30903	0.19
		0.10	0.26726			
0.20	0.00	0.00	0.90807	0.79395	0.92699	12.57
		0.10	0.92460			
	0.01	0.00	0.84133	0.76210	0.85897	9.42
		0.10	0.85346			
	0.02	0.00	0.78549	0.73279	0.81958	6.71
		0.10	0.80755			
	0.05	0.00	0.67172	0.65782	0.73300	2.07
		0.10	0.70532			
	0.10	0.00	0.56944	0.55884	0.62859	1.92
		0.10	0.58915			
	0.20	0.00	0.41943	0.41489	0.47797	1.08
		0.10	0.43348			
	0.40	0.00	0.22918	0.22726	0.27285	0.83
		0.10	0.23680			

It should be noted that the preceding perfect ($\theta_{10} = \theta_{20} = 0$) model under static loading exhibits an unstable symmetric branching point with corresponding critical load $\lambda_c = 1$. For an imperfect system with axisymmetric imperfections ($\theta_{10} \neq 0$, $\theta_{20} = 0$), instability takes place via an unstable branching point lying on a nontrivial fundamental path with corresponding loading λ_{cc}. A large variety of numerical results corresponding to nonsymmetric as well as to symmetric systems (imperfect with axisymmetric imperfections) are given in graphical and tabular form.

Figure 4 illustrates the variation of the angles θ_1 and θ_2 vs τ corresponding to unbounded motions near the instant of dynamic buckling for an undamped system, with $\theta_{10} = 0.10$ and $\theta_{20} = 0.01$ under step loading of infinite duration. From Fig. 5, one can see the variation of $\dot{\theta}_1$ and $\dot{\theta}_2$ vs τ corresponding to the previous case. The unbounded motions of this system in the

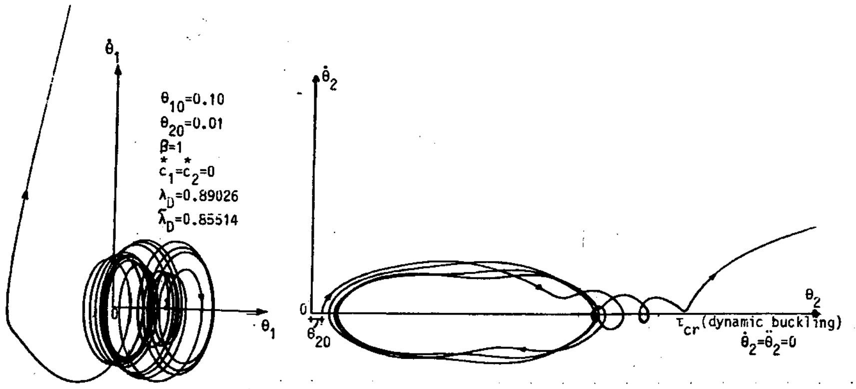

Fig. 6 Unbounded (Θ_1, $\dot{\Theta}_1$)- and (Θ_2, $\dot{\Theta}_2$)-phase plane motions (near dynamic buckling) of a nonsymmetric undamped system.

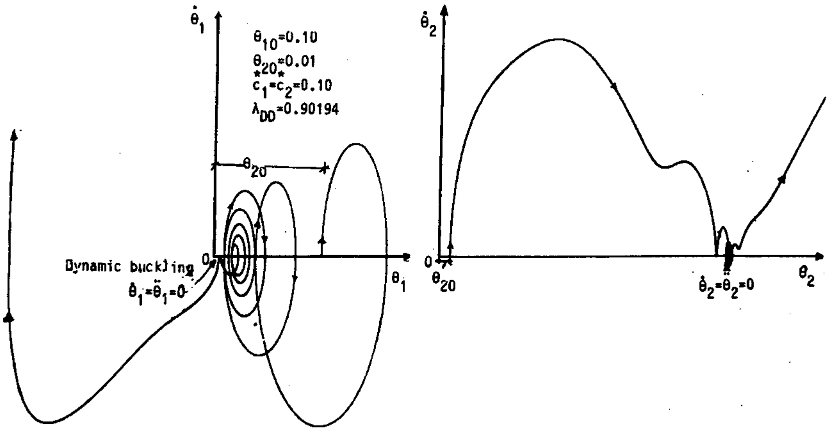

Fig. 7 Unbounded (Θ_1, $\dot{\Theta}_1$)- and (Θ_2, $\dot{\Theta}_2$)-phase plane motions (near dynamic buckling) of a nonsymmetric damped system.

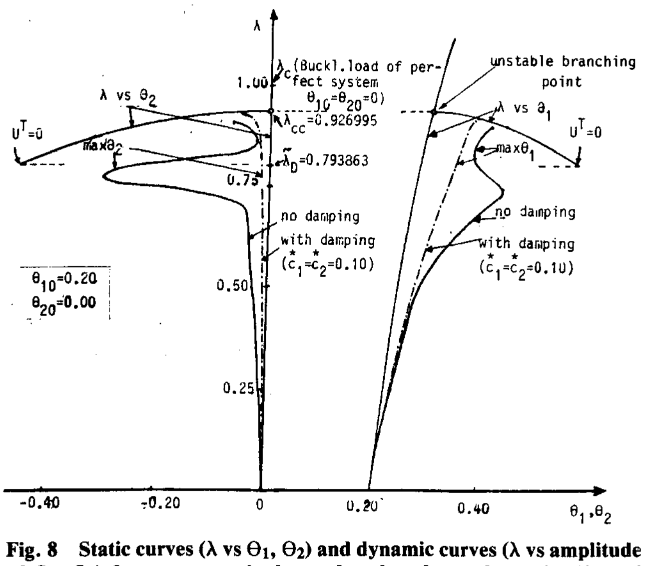

Fig. 8 Static curves (λ vs Θ_1, Θ_2) and dynamic curves (λ vs amplitude of Θ_1, Θ_2) for a symmetric damped and undamped step loading of infinite duration.

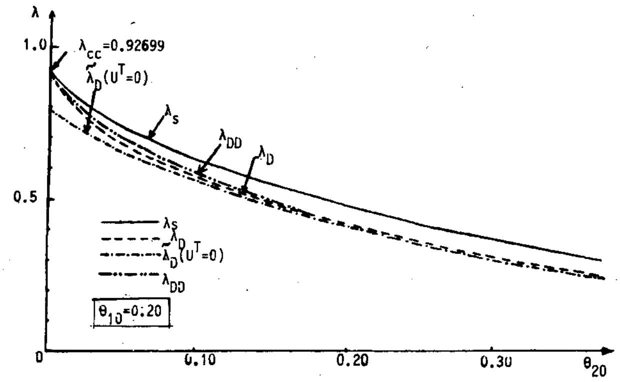

Fig. 9 Static buckling load λ_s and dynamic buckling loads λ_D and λ_{DD} for a system with $\Theta_{10} = 0.20$ and varying Θ_{20}.

(θ_1, $\dot{\theta}_1$)- and (θ_2, $\dot{\theta}_2$)-phase planes without or with damping are shown in Figs. 6 and 7. From Figs. 4–7, it can be easily seen that the instability criterion (19) is satisfied. Note that, if there is no damping, the inflection point criterion is satisfied only by θ_1, whereas if damping is included it seems to be satisfied by both angles.

From Table 1, one can see the change of the critical loads, static λ_{cc} (or λ_s), exact dynamic λ_D (or λ_{DD}) and approximate $\tilde{\lambda}_D$ for a variety of symmetric and nonsymmetric systems with or without damping under step loading of infinite duration. From the last column of this table it can be seen that the difference $\lambda_D - \tilde{\lambda}_D$ decreases considerably as the system goes away from its symmetrical configuration; in the extreme case, the maximum difference in these loads is 12.57%. However, in such a case, instead of $\tilde{\lambda}_D$ one can adopt λ_{cc}, which is slightly higher than λ_D and almost coincident with λ_{DD}. Moreover, it is apparent that the imperfection sensitivity under dynamic loading is more pronounced than that under static loading. Indeed, if θ_{10} is kept constant, λ_D decreases more rapidly than λ_{cc} (or λ_s) as θ_{20} increases. The inclusion of damping increases slightly the dynamic buckling load λ_{DD}, which approaches λ_{cc}. In the extreme case, the difference between λ_{cc} and λ_{DD} is less than 0.25%. It is also observed that inequality (26) is satisfied.

From the plot in Fig. 8, corresponding to a symmetric system with $\theta_{10} = 0.20$ and $\theta_{20} = 0$, one can see the static post-buckling equilibrium paths (λ vs θ_1 and θ_2) traced to $U_T = 0$ and the dynamic curves (λ_D or λ_{DD} vs amplitude of θ_1 and θ_2).

Figure 9 illustrates the variation of λ_D, λ_{DD}, λ_s, and of the lower bound buckling load $\tilde{\lambda}_D$ for varying θ_{20} and fixed $\theta_{10}(=0.20)$.

It is worth observing that dynamic buckling of imperfect systems with axisymmetric imperfections may occur at large time, particularly as these systems approach the perfect one ($\theta_{10}=\theta_{20}=0$). However, when $\theta_{10}<0.20$ one can overcome this drawback by adopting the slightly higher dynamic buckling loads λ_{cc}, which can be obtained via a static stability analysis.

In conclusion, the inflection point dynamic instability criterion is valid for limit point systems, including those generated from bifurcational systems with branching points lying on nontrivial fundamental paths. A typical example is illustrated in Fig. 10.

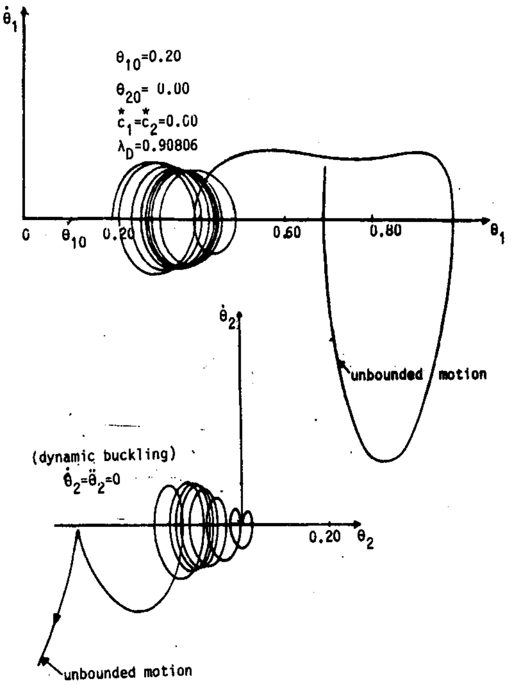

Fig. 10 Unbounded motions in the (Θ_1, $\dot{\Theta}_1$)- and (Θ_2, $\dot{\Theta}_2$)-phase planes of an undamped symmetric system with axisymmetric imperfections.

Model 2

The second model is the double pendulum shown in Fig. 11 that consists of two rigid weightless bars of equal lengths ℓ, interconnected with each other and with the support by frictionless hinges. Two concentrated masses m_1 and m_2 are placed at B and A. The model is subjected to a vertical step load P of infinite duration applied at its tip. The configuration of this two degree-of-freedom model is completely specified by the angles θ_1 and θ_2 (with respect to the vertical position of each of the bars) and their angular velocities $\dot{\theta}_1$ and $\dot{\theta}_2$, respectively. Structural stiffness is provided by two nonlinear torsional springs located at B and C with restoring dimensionless moments

$$\frac{M_c}{\kappa}=\vartheta_1-\epsilon_1+\delta_1(\vartheta_1-\epsilon_1)^2+\gamma_1(\vartheta_1-\epsilon_1)^3 \tag{30a}$$

$$\frac{M_B}{\kappa}=\vartheta_2-\epsilon_2-\vartheta_1+\epsilon_1+\delta_2(\vartheta_2-\epsilon_2-\vartheta_1+\epsilon_1)^2$$
$$+\gamma_2(\vartheta_2-\epsilon_2-\vartheta_1+\epsilon_1)^3 \tag{30b}$$

where the unrestrained configuration is identified by the initial imperfections $\vartheta_1=\epsilon_1$ and $\vartheta_2=\epsilon_2$; κ is the linear spring component, and δ_i, $\gamma_i(i=1,2)$ the nonlinear spring components. For $\gamma_1=\gamma_2=0$ and δ_1, $\delta_2\neq 0$ we have a nonlinear elastic quadratic model, whereas, if $\delta_1=\delta_2=0$ and γ_1, $\gamma_2\neq 0$, the model becomes a nonlinear elastic cubic one. When all nonlinear spring components are negative (positive), the nonlinear elastic material is of "soft" ("hard") type. Regarding the stability of the perfect system under a statically applied load, it can be shown that, if $\gamma_1=\gamma_2=0$ and $\delta_1=\delta_2$ are not both zero, the model is associated with a symmetric branching point whose stability depends on the values of γ_1 and γ_2. One could also include the effect of linear viscous damping by adding to the restoring moments M_c and M_B the terms $c_1\dot{\vartheta}_1$ and $c_2(\dot{\vartheta}_2-\dot{\vartheta}_1)$, respectively.

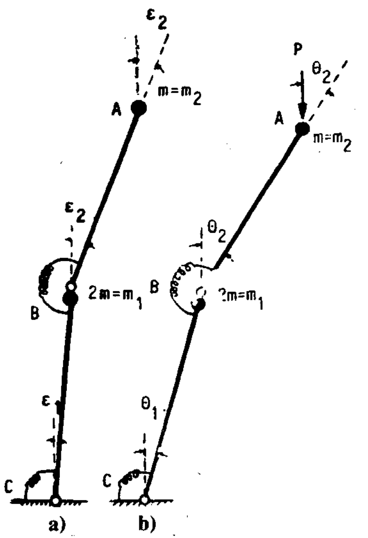

Fig. 11 Unstressed a) and stressed b) state of a double pendulum under step loading.

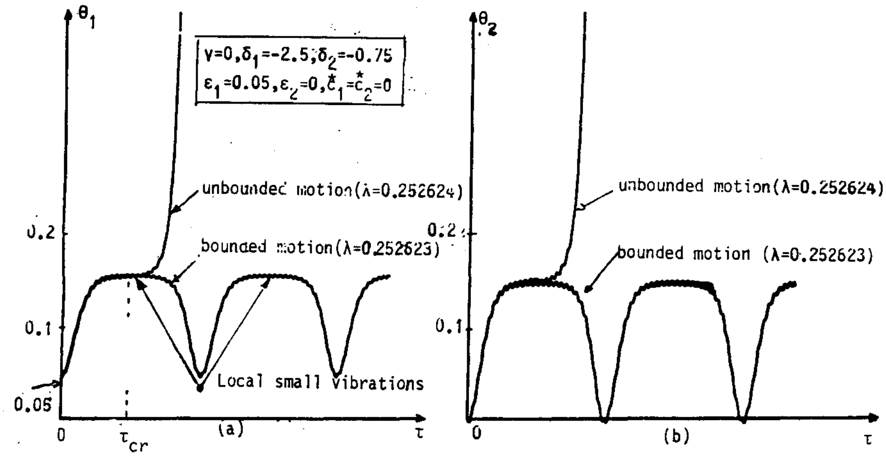

Fig. 12 Dynamic curves Θ_1 and Θ_2 vs τ of bounded and unbounded motion for an undamped system under step loading of infinite duration.

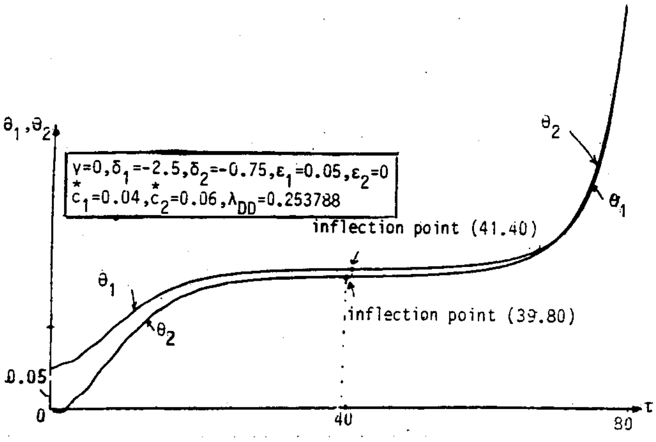

Fig. 13 Unbounded motions Θ_1 and Θ_2 vs τ when slight damping is included.

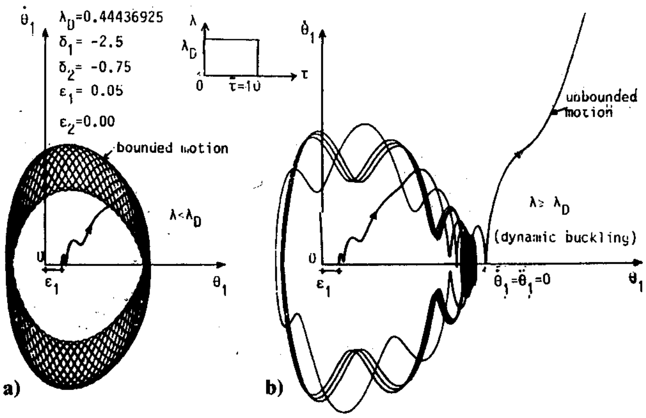

Fig. 14 Motions in the $(\Theta_1, \dot{\Theta}_1)$-phase planes of an undamped system under a step rectangular loading: a) bounded and b) unbounded.

The kinetic energy K, the potential energy U_T, and the Rayleigh dissipation function F are

$$K = \tfrac{1}{2} m_1 \ell^2 \dot{\vartheta}_1^2 + \tfrac{1}{2} m_2 \ell^2 [\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1 \dot{\vartheta}_2 \cos(\vartheta_1 - \vartheta_2)] \quad (31a)$$

$$\frac{U_T}{\kappa} = \tfrac{1}{2}(\vartheta_1 - \epsilon_1)^2 + \tfrac{1}{3}\delta_1(\vartheta_1 - \epsilon_1)^3 + \tfrac{1}{4}\gamma_1(\vartheta_1 - \epsilon_1)^4$$
$$+ \tfrac{1}{2}(\vartheta_2 - \epsilon_2 - \vartheta_1 + \epsilon_1)^2 + \tfrac{1}{3}\delta_2(\vartheta_2 - \epsilon_2 - \vartheta_1 + \epsilon_1)^3$$
$$+ \tfrac{1}{4}\gamma_2(\vartheta_2 - \epsilon_2 - \vartheta_1 + \epsilon_1)^4 - \lambda(\cos\epsilon_1 - \cos\vartheta_1$$
$$+ \cos\epsilon_2 - \cos\vartheta_2) \qquad \text{with } \lambda = P\ell/\kappa \quad (31b)$$

$$F = \tfrac{1}{2} c_1 \dot{\vartheta}_1^2 + \tfrac{1}{2} c_2 (\dot{\vartheta}_2 - \dot{\vartheta}_1)^2 \quad (31c)$$

Introducing the nondimensionalized quantities

$$\theta(\tau) = \vartheta(t), \qquad \tau = t\sqrt{\kappa/m_2\ell^2}, \qquad m = m_1/m_2$$
$$\bar{c}_i = \frac{c_i}{\ell\sqrt{\kappa m_2}} \qquad (i = 1,2) \quad (32)$$

Equations (1) by virtue of relations [Eqs. (30) and (31)] give

$$(1 + m)\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$
$$+ (\bar{c}_1 + \bar{c}_2)\dot{\theta}_1 - \bar{c}_2\dot{\theta}_2 + \frac{\partial V_T}{\partial \theta_1} = 0 \quad (33a)$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \bar{c}_2\dot{\theta}_2 - \bar{c}_2\dot{\theta}_1 + \frac{\partial V_T}{\partial \theta_2} = 0 \quad (33b)$$

where

$$\frac{\partial V_T}{\partial \theta_1} = 2(\theta_1 - \epsilon_1) - (\theta_2 - \epsilon_2) + \delta_1(\theta_1 - \epsilon_1)^2$$
$$+ \gamma_1(\theta_1 - \epsilon_1)^3 - \delta_2(\theta_1 - \epsilon_1 - \theta_2 + \epsilon_2)^2$$
$$+ \gamma_2(\theta_1 - \epsilon_1 - \theta_2 + \epsilon_2)^3 - \lambda \sin\theta_1 \quad (34a)$$

$$\frac{\partial V_T}{\partial \theta_2} = -(\theta_1 - \epsilon_1) + (\theta_2 - \epsilon_2) + \delta_2(\theta_1 - \epsilon_1 - \theta_2 + \epsilon_2)^2$$
$$-\gamma_2(\theta_1 - \epsilon_1 - \theta_2 + \epsilon_2)^3 - \lambda \sin\theta_2 \quad (34b)$$

A variety of numerical results in graphical form are shown in Figs. 12–14 for an imperfect system, corresponding to $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\gamma_1 = \gamma_2 = \gamma = 0$, $\epsilon_1 = 0.05$, and $\epsilon_2 = 0$, which loses its static stability via snapping at a critical load $\lambda_s = 0.267713$. Dynamic buckling under step loading of infinite duration occurs at a load $\lambda_D = 0.252624$ (i.e., lower than λ_s by about 6%). From Figs. 12a and 12b one can see the corresponding variations of $\theta_1(\tau)$ and $\theta_2(\tau)$ vs τ for a bounded $(\lambda < \lambda_D)$, as well as for an unbounded $(\lambda > \lambda_D)$ motion. One can also observe localized small amplitude vibrations of very low period in addition to the overall oscillatory motion of the system. In this case, the inflection point instability criterion (19) is satisfied by the averages of $\theta_1(\tau)$ and $\theta_2(\tau)$. The inclusion of slight damping ($\bar{c}_1 = 0.04$, $\bar{c}_2 = 0.06$) in this system yields the motions θ_1 and θ_2 vs τ, shown in Fig. 13, which correspond to the dynamic buckling load $\lambda_{DD} = 0.253788 < \lambda_s = 0.267713$. The localized small vibrations of the undamped system are not any more visible due to the inclusion of damping; hence, the curves θ_1 and θ_2 vs τ are sufficiently smooth to allow the application of the dynamic instability criterion, [Eq. (19)]. From this model, one can draw the important conclusion that the instability criterion based on the average curves $\theta_1(\tau)$ and $\theta_2(\tau)$ extends criterion (19).

Application of the static criterion in Eq. (20) yields an approximate buckling load for an undamped system equal to $\bar{\lambda}_D = 0.247515$; that is 2% smaller than the "exact" $\lambda_D = 0.252624$.

For an imperfect system with $\delta_1 = \delta_2 = 0$, $\gamma = -0.25$, $\epsilon_1 = 0.05$, $\epsilon_2 = 0$, we find $\lambda_s = 0.4042819$ with $|\max(\theta_1, \theta_2)| < 1.40$, $\lambda_D = 0.390583$ with $|\max(\theta_1, \theta_2)| < 1.65$, whereas $\bar{\lambda}_D = 0.390569$ with $|\max(\theta_1, \theta_2)| < 1.64$. That is, the approximate buckling load $\bar{\lambda}_D$ practically coincides with the exact λ_D. It is generally observed from the results of the second model that the velocities $\dot{\theta}_1$ and $\dot{\theta}_2$ vanish almost simultaneously at the instant of dynamic buckling. This explains why $U_T = 0$ yields dynamic buckling loads $\bar{\lambda}_D$ very close to the exact λ_D.

The response of an undamped system corresponding to $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\epsilon_1 = +0.05$, $\epsilon_2 = 0$ under step loading of rectangular shape with $\bar{\tau} = 10$ is shown in Fig. 14. It can be clearly seen that the dynamic stability criterion (19) is also satisfied for this case of loading.

Another remarkable result is obtained for an imperfect structural system with $\gamma = 0$, $\delta_1 = -2.5$, $\delta_2 = -0.75$, $\epsilon_1 = 0.05$, and $\epsilon_2 = -0.032$. This system is stable under static loading (exhibiting a continuous rising path) for $\epsilon_2 < -0.032$ but dynamically unstable under step loading of infinite duration. This important finding, reported for the first time in technical literature, has the following explanation.

The aforementioned stable system, under static loading, in addition to its continuously rising (stable) equilibrium path, experiences a complementary unstable equilibrium path.[28] Numerical simulation reveals that dynamic buckling (escaped motion) may be initiated from singular points of the complementary path or nonsingular points in the neighborhood of this path. This can be achieved by examining the eigenvalues of the Jacobian matrix [see Eqs. (9) and (10)]. From a theoretical viewpoint, one can show that dynamic instability is possible, since the corresponding Jacobian matrix (evaluated at equilibrium points of the complementary path) has at least one eigenvalue with positive real part.

With the aid of transformation $\theta_1 = y_1$, $\dot{\theta}_1 = y_2$. $\theta_2 = y_3$, and $\dot{\theta}_2 = y_4$, Eqs. (33) can be written in the form of Eqs. (5) as a system of four first-order differential equations

$$\dot{y}_i = Y_i \qquad (i = 1,2,3,4) \tag{35}$$

where

$$Y_1 = y_2, Y_2 = \frac{1}{m + \sin^2(y_1 - y_3)}\Bigg[\bar{c}_2[\cos(y_1 - y_2) + 1]y_4 - \{\bar{c}_1 + \bar{c}_2[1 + \cos(y_1 - y_3)]\}y_2 - \frac{y_2^2}{2}\sin 2(y_1 - y_3) - y_4^2 \sin(y_1 - y_3) - \frac{\partial V_T}{\partial y_1} - \frac{\partial V_T}{\partial y_3}\cos(y_1 - y_3)\Bigg], \qquad Y_3 = y_4 \tag{36a}$$

$$Y_4 = \frac{1}{m + \sin^2(y_1 - y_3)}\Bigg[[(m+1)\bar{c}_1 + (\bar{c}_1 + \bar{c}_2)\cos(y_1 - y_3)]y_2 - [\bar{c}_2(m+1) + \bar{c}_2\cos(y_1 - y_3)]y_4 + (m+1)y_2^2 \sin(y_1 - y_3) + \frac{y_4^2}{2}\sin 2(y_1 - y_3) + \frac{\partial V_T}{\partial y_1}\cos(y_1 - y_3) - (m_1 + 1)\frac{\partial V_T}{\partial y_3}\Bigg] \tag{36b}$$

By virtue of Eqs. (36), the Jacobian matrix of system (35) evaluated at an equilibrium state y^E of the complementary path is given by

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial Y_2(y^E)}{\partial y_1} & \frac{\partial Y_2(y^E)}{\partial y_2} & \frac{\partial Y_2(y^E)}{\partial y_3} & \frac{\partial Y_2(y^E)}{\partial y_4} \\ 0 & 0 & 0 & 1 \\ \frac{\partial Y_4(y^E)}{\partial y_1} & \frac{\partial Y_4(y^E)}{\partial y_2} & \frac{\partial Y_4(y^E)}{\partial y_3} & \frac{\partial Y_4(y^E)}{\partial y_4} \end{bmatrix} \tag{37}$$

It can be shown that the characteristic equation $|Y_y(y^E) - \rho I| = 0$ of matrix (37) leads to a polynomial of the form

$$\rho^4 + \alpha_1\rho^3 + \alpha_2\rho^2 + \alpha_3\rho + \alpha_4 = 0 \tag{38}$$

where

$$\alpha_4 = \left(\frac{\partial Y_2}{\partial y_1}\cdot\frac{\partial Y_4}{\partial y_3} - \frac{\partial Y_2}{\partial y_3}\cdot\frac{\partial Y_4}{\partial y_1}\right)_{y=y^E} = \frac{1}{m + \sin^2(y_1 - y_3)}\left[\frac{\partial^2 V_T}{\partial y_1^2}\cdot\frac{\partial^2 V_T}{\partial y_3^2} - \left(\frac{\partial^2 V_T}{\partial y_1 \partial y_3}\right)^2\right]_{y=y^E}$$

is the determinant of the Jacobian. The sign of α_4 depends on the sign of the quantity between brackets, which is related to the second variation $\delta^2 V_T$. This quantity is negative when $\delta^2 V_T(/y^E)$ is evaluated at the unstable complementary equilibrium path; hence, $\alpha_4 < 0$. As is known from the theory of equations, every algebraic equation of even degree whose constant term is negative has at least two real roots, one positive and the other negative. Then the Jacobian evaluated at y^E is associated with a saddle point from which (or from its neighborhood) incipient dynamic buckling may occur, a situation presently numerically verified.

Hence, dynamic buckling under step loading of infinite duration may also occur in another class of systems besides limit point systems. This class includes statically stable systems that also display a complementary unstable equilibrium path.

It is worthy to note that despite the availability of high-speed computer and modern computational techniques, the solution of the original, highly nonlinear, differential equations of motion is quite often associated with serious numerical difficulties. This drawback is more acute in problems of nonlinear dynamic stability, in which an unbounded motion may be initiated after a very long period of time. This quite often is due to chaotic phenomena that may be present in Hamiltonian systems.[29] In view of these very serious computational difficulties, the importance of establishing lower bound dynamic buckling estimates (which can be readily obtained) is evident.

Finally, it should be noted that the preceding analysis can be applied by structural engineers whose mathematical training does not extend beyond the classical methods of analysis. However, if one employed a more strict mathematical way, many of the above findings could be established very conveniently and readily by using the theory of attractors and differentiable manifolds.[30]

Conclusions

This investigation deals with general damped or undamped limit point bifurcational (with trivial or nontrivial primary paths) systems under step loading of infinite or finite duration. The most important findings—based on a large variety of results corresponding to two different models with two degrees of freedom—are the following:

1) The dynamic instability mechanism and the nature of the dynamic critical points via which dynamic buckling (escaped motion) may be initiated are thoroughly discussed in light of the global response.

2) Dynamic instability criteria are established according to which dynamic buckling occurs when at least one of the generalized coordinates vs time (or its average value) exhibits an inflection point.

3) Dynamic buckling loads of bifurcational systems with trivial fundamental paths coincide with the corresponding static buckling loads. Bifurcational systems with nontrivial fundamental paths are associated with dynamic buckling loads almost coincident with the corresponding static ones.

4) The dynamic critical point lies on the secondary unstable path in case of one-degree-of-freedom damped or undamped systems; if there is no damping, such a point is a singular point with zero total potential energy. For two-mass undamped systems, the dynamic critical point is a regular point close to the unstable equilibrium path.

5) The foregoing findings allow us to establish a static instability criterion associated with zero total potential energy, which for systems under step loading of infinite duration yields: a) exact dynamic buckling loads for one-degree-of-freedom undamped systems and lower bound dynamic buckling estimates if damping is included, b) lower bound buckling estimates for general two-degree-of-freedom damped or undamped systems, c) exact dynamic buckling loads only when the branching point lies on a trivial fundamental path. On the contrary, if the branching point lies on a nontrivial fundamental path (imperfect systems with axisymmetric imperfections), U_T is different from zero at that point; however, U_T vanishes at an equilibrium point on the secondary unstable path with corresponding load $\bar{\lambda}_D$ less than λ_{cc}. The maximum difference $\lambda_{cc} - \bar{\lambda}_D$ in the symmetric systems under discussion is about 12.5%, but it decreases considerably as the systems become gradually nonsymmetric; then the difference $\lambda_s - \lambda_D$ decreases with decreasing limit point load.

6) The maximum possible error between the exact dynamic buckling load λ_D or λ_{DD} and the lower bound buckling esti-

mate $\tilde{\lambda}_D$ is much less than that between the corresponding static limit point load λ_s and $\tilde{\lambda}_D$. In the extreme case of symmetric systems with symmetric imperfections, there is no need to apply the static stability criterion since the exact dynamic buckling load λ_D (or λ_{DD}) is either very close to the bifurcational load λ_{cc} or practically coincident with it in case of small axisymmetric imperfections. Hence, the static stability criterion yields very reliable dynamic buckling loads for general nonsymmetric and symmetric systems.

7) The effect of imperfection sensitivity is more severe under dynamic step loading, as compared to the static loading.

8) Statically stable systems exhibiting a complementary unstable equilibrium path may become dynamically unstable under step loading of infinite duration.

9) Solutions should be discussed very carefully, due to the likely effects of chaotic phenomena that quite often are manifested after a long period of apparent quiescence.

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